# Phase transitions in a generalised two-dimensional Ising gauge plus matter theory



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# Phase transitions in a generalised two-dimensional Ising gauge plus matter theory

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**Abstract.** The two-dimensional version of a generalised Ising gauge theory in which the gauge part of the action involves a product of Ising link variables around every rectangular $2\times1$ face of the lattice is studied. The pure gauge theory is related to the Ising model and is self-dual. The gauge plus matter theory is also self-dual. For negative gauge coupling an Ising critical line in the $(\beta_g, \beta_m)$ plane separates a phase with staggered frustration from another which is homogeneously frustrated.

## 1. Introduction

Gauge theories in $D$ dimensions are believed to be closely related to spin theories with the same symmetry group in $D/2$ dimensions. The lower critical dimensionality is $D_c = 1$ for spin theories and $D_c = 2$ for gauge theories with discrete symmetry groups whereas it is 2 and 4 for continuous symmetry groups (Wegner 1971, Balian *et al* 1975, Kogut 1979, Toulouse 1980, Pearson 1981). On hypercubical lattices the Ising pure gauge theory is self-dual in 4D whereas the spin problem is self-dual in 2D (Wannier and Kramers 1941). In 3D the gauge plus matter theory is self-dual and the pure gauge theory is related by duality to the Ising model. In 2D the pure gauge theory is trivial and the gauge plus matter theory is related to the Ising model in an external field. As a consequence there is no transition for finite values of the gauge coupling.

All these results were obtained for a pure gauge part of the action involving four link variables around an elementary interaction loop (plaquette). Generalised lattice gauge actions involving larger interaction loops were recently proposed and studied in 3D and 4D (Edgar 1982, Bhanot *et al* 1983). In the present work we look at the 2D version of one of these generalised lattice gauge actions involving the product of six Ising link variables around every rectangular face (window in the terminology of Bhanot *et al* (1983)) of the lattice.

The action is given by

$$S(\beta_m, \beta_g) = \beta_m \sum_{(ij)} \sigma_i U_{ij} \sigma_j + \beta_g \sum_{\square\square} U_{\square\square}. \tag{1.1}$$

The matter part, with coupling $\beta_m$, involves Ising spins $\sigma = \pm 1$ on the $N$ sites and Ising gauge variables $U_{ij}$ on the $2N$ links of the square lattice. The gauge part, with

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coupling $\beta_g$, is a sum over the $2N$ windows (there is a one-to-one correspondence between windows and dual links) of gauge variable products (figure 1)

$$U_{\square\square} = U_{ij}U_{jk}U_{kl}U_{lm}U_{mn}U_{ni}. \tag{1.2}$$

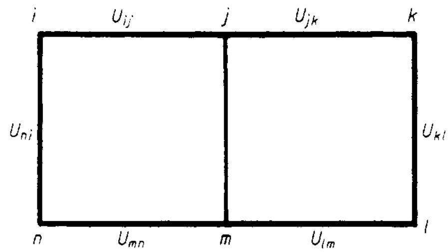


**Figure 1.** The gauge part of the generalised action involves a product of six link variables around every window.

The action is invariant under Ising gauge transformations:

$$\sigma_i \to s_i\sigma_i, \qquad U_{ij} \to A_{ij} = s_i U_{ij} s_j, \tag{1.3}$$

where $s_i$ is an Ising variable associated with site $i$. With $s_i = -1$ and $s_j = +1\ \forall j \neq i$, we get a local gauge transformation

$$\sigma_i \to -\sigma_i, \qquad U_{ij} \to -U_{ij}\ \forall j, \tag{1.4}$$

under which (1.1) is invariant. The partition function is also invariant under the change $\beta_m \to -\beta_m$, since changing $\sigma_i$ into $-\sigma_i$ on one of the two square sublattices gives back positive couplings on every link. In the following we shall restrict ourselves to the positive sector ($\beta_m > 0$) of the phase diagram.

The outline of the paper is as follows: in § 2 the pure gauge theory ($\beta_m = 0$) is related to the Ising model and as a consequence is self-dual whereas the mixed gauge theory

$$S(\beta_p, \beta_g) = \beta_p \sum_{\square} U_{\square} + \beta_g \sum_{\square\square} U_{\square\square}, \tag{1.5}$$

where

$$U_{\square} = U_{ij}U_{jk}U_{kl}U_{li} \tag{1.6}$$

is the usual gauge interaction involving four links around a plaquette, is related to the Ising model in an external field. The gauge plus matter theory is shown to be self-dual in § 3. In § 4 we show that the pure gauge critical point is unstable when a weak matter field is introduced. In § 5 the $\beta_g < 0$ sector of the generalised gauge plus matter theory is examined.

## 2. Pure gauge and mixed action theories: relation with the Ising model

The partition function of the pure gauge model (equation (1.1) with $\beta_m = 0$) reads

$$Z(\beta_g) = 2^N \sum_{\{U\}} \prod_{\square\square} \exp(\beta_g U_{\square\square}) \tag{2.1}$$

where the factor $2^N$ comes from the sum over the non-interacting spins $\{\sigma\}$. The modified gauge interaction $U_{\square\square}$ may be rewritten as the product of two nearby frustrations $U_\square = \pm 1$ (Toulouse 1977). Introducing Ising spins $\mu_\alpha$ on the sites of the dual lattice such that

$$\mu_\alpha = U_\square \tag{2.2}$$

the partition function becomes

$$Z(\beta_g) = 2^{2N} Z_{\text{Ising}}(\beta_g) \tag{2.3}$$

where

$$Z_{\text{Ising}}(\beta_g) = \sum_{\{\mu\}} \prod_{(\alpha\beta)} \exp(\beta_g \mu_\alpha \mu_\beta) \tag{2.4}$$

is the Ising model partition function. A factor $2^N$ has been included in equation (2.3) to take into account the gauge invariance of the $\{\mu\}$ configurations. The relation with the Ising model may be also obtained through a duality transformation on the pure gauge model.

As a consequence of this relation the pure gauge theory is self-dual (the self-duality may be obtained directly on the pure gauge model) and has an Ising critical point at

$$\beta_g = \beta_c = \tfrac{1}{2}\ln(1+\sqrt{2}). \tag{2.5}$$

Besides the local $Z_2$ gauge symmetry, the pure gauge action is invariant under a global frustration reversal ($U_\square \to -U_\square$ on every plaquette) and the second-order phase transition does not contradict Elitzur's theorem (Elitzur 1975) since it is this global symmetry which is broken when $\beta_g \geq \beta_c$. The frustration, a gauge invariant quantity, is the order parameter. The average frustration is zero below $\beta_c$ and non-vanishing above:

$$\langle U_\square \rangle \sim (\beta_g - \beta_c)^{1/8} \qquad (\beta_g \geq \beta_c). \tag{2.6}$$

Since the transition has nothing to do with the local gauge invariance, the Wilson loop correlation function (Wegner 1971, Wilson 1974)

$$G(\Gamma) = \left\langle \prod_\Gamma U \right\rangle \tag{2.7}$$

which is the thermal average of the product of link variables around a loop $\Gamma$, follows an area law both above and below $\beta_c$:

$$G(\Gamma) \sim \exp(-\mu A_\Gamma) \tag{2.8}$$

where $A_\Gamma$ is the loop area.

This is easily verified through low- and high-temperature expansions. Using the correspondence with the Ising model equation (2.7) may be rewritten as the average of the product of Ising spins inside the loop:

$$G(\Gamma) = \left\langle \prod_{\alpha \subset A_\Gamma} \mu_\alpha \right\rangle \tag{2.9}$$

and the leading term in the high-temperature expansion is given by all possible pairings of the $A_\Gamma$ interior spins with nearest neighbour bonds. This is nothing else than the dimer problem on the loop (Kasteleyn 1961, Temperley and Fisher 1961). Since $A_\Gamma/2$

dimers are required, we get

$$G(\Gamma) \simeq \kappa^{A_\Gamma} (\tanh \beta_g)^{A_\Gamma/2} \tag{2.10}$$

where $\kappa$ is the dimer partition function per site. Using standard methods, the leading term in the low-temperature expansion reads

$$G(\Gamma) \simeq \exp[-2 \exp(-8\beta_g) A_\Gamma]. \tag{2.11}$$

Let us now consider the mixed action $S(\beta_p, \beta_g)$ (equation (1.5)) which is the 2D version of the action studied by Bhanot *et al* (1983). Using the dual Ising spin representation of the frustration (equation (2.2)), the partition function of the mixed action model may be written as

$$Z(\beta_p, \beta_g) = 2^N \sum_{\{\mu\}} \exp\left(\beta_p \sum_\alpha \mu_\alpha + \beta_g \sum_{(\alpha\beta)} \mu_\alpha \mu_\beta\right) \tag{2.12}$$

so that the mixed action model is equivalent to an Ising model with nearest neighbour interaction $\beta_g$ in an external field $\beta_p$ breaking the global spin reflection symmetry. This correspondence will be useful in the study of the stability of the Ising critical point in the gauge plus matter theory (§ 4).

## 3. Self-duality of the gauge plus matter theory

The partition function of the gauge plus matter theory is invariant under the unitary gauge transformation (equation (1.3) with $s_i = \sigma_i$) and reads

$$Z(\beta_m, \beta_g) = \sum_{\{\sigma\}} \sum_{\{A\}} \exp[S(\beta_m, \beta_g)] = 2^N \sum_{\{A\}} \prod_{(ij)} \exp(\beta_m A_{ij}) \prod_{\square\square} \exp(\beta_g A_{\square\square}). \tag{3.1}$$

Introducing new variables $t_m$ and $t_g = 0, 1$ associated with links and windows and using the identity (Savit 1980)

$$\exp \beta A = \sum_{t=0,1} C_t(\beta) A^t, \qquad t = t_m, t_g;\ \beta = \beta_m, \beta_g;\ A = A_{ij}, A_{\square\square}, \tag{3.2}$$

where

$$C_t(\beta) = \cosh \beta \exp[t \ln(\tanh \beta)] \tag{3.3}$$

we get

$$\begin{aligned} Z(\beta_m, \beta_g) &= 2^N \sum_{\{t_m\}} \sum_{\{t_g\}} \prod_{(ij)} C_{t_m}(\beta_m) \prod_{\square\square} C_{t_g}(\beta_g) \sum_{\{A\}} \prod_{(ij)} A_{ij}^{t_m} \prod_{\square\square} A_{\square\square}^{t_g} \\ &= 2^N \sum_{\{t_m\}} \sum_{\{t_g\}} \prod_{(ij)} C_{t_m}(\beta_m) \prod_{\square\square} C_{t_g}(\beta_g) \prod_{(ij)} \sum_{A_{ij}=\pm 1} A_{ij}^{t_m + \sum t_g} \end{aligned} \tag{3.4}$$

where $t_m + \Sigma t_g$ are the $t$ variables for links and windows involving the link variable $A_{ij}$ (figure 2($a$)). Summing over $A_{ij}$ for each link $(ij)$,

$$Z(\beta_m, \beta_g) = 2^{3N} \sum_{\{t_m\}} \sum_{\{t_g\}} \prod_{(ij)} C_{t_m}(\beta_m) \prod_{\square\square} C_{t_g}(\beta_g) \prod_{(ij)} \delta_2\left(t_m + \sum t_g\right). \tag{3.5}$$

The Kronecker delta functions are automatically satisfied by the following representation of the $t$ variables (figure 2($b$)):

$$t_m = \tfrac{1}{2}(1 - V_{\square\square}), \tag{3.6}$$

$$t_g = \tfrac{1}{2}(1 - V_{ij}), \tag{3.7}$$

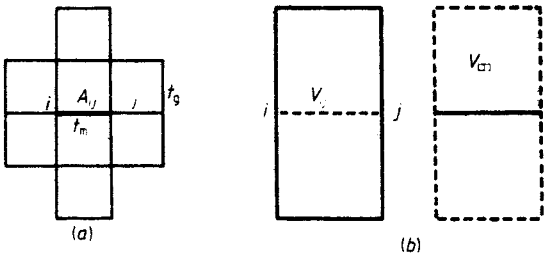


**Figure 2.** (*a*) Windows and links involving the link variable $A_{ij}$. (*b*) Representation of the $t_m$ and $t_g$ variables.

where $V_{ij}$ is an Ising link variable and $V_{\square\square}$ a product of $V$ link variables around a window of the original lattice. Then, using equation (3.3),

$$Z(\beta_m, \beta_g) = (\sinh 2\beta_m \sinh 2\beta_g)^N 2^N \sum_{\{V\}} \exp\left(\tilde{\beta}_m \sum_{(ij)} V_{ij} + \tilde{\beta}_g \sum_{\square\square} V_{\square\square}\right)$$
$$= (\sinh 2\beta_m \sinh 2\beta_g)^N Z(\tilde{\beta}_m, \tilde{\beta}_g). \tag{3.8}$$

The dual couplings satisfy

$$\exp(-2\tilde{\beta}_m) = \tanh \beta_g, \tag{3.9}$$

$$\exp(-2\tilde{\beta}_g) = \tanh \beta_m. \tag{3.10}$$

The model is self-dual; links and windows are exchanged in the duality transformation. The line $\beta_m = 0$ is transformed into the line $\beta_g = \infty$ and the pure gauge critical point $(\beta_g = \beta_c, \beta_m = 0)$ gives a new critical point $(\beta_g = \infty, \beta_m = \beta_c)$.

When $\beta_g = \infty$ the gauge degrees of freedom are frozen into a state with $U_{\square\square} = 1$ on each window. In terms of frustration variables (figure 1)

$$U_{\square\square}(ijklmn) = U_{\square}(ijmn) U_{\square}(jklm) \tag{3.11}$$

so that the constraints may be satisfied either with a fully frustrated state ($U_{\square} = -1\ \forall\square$) or with an unfrustrated state ($U_{\square} = +1\ \forall\square$). The matter part of the action favours the unfrustrated state which is realised for all non-vanishing $\beta_m$. Choosing a gauge where all the links are in the state $U_{ij} = +1$, we get a ferromagnetic Ising model and the point $(\beta_g = \infty, \beta_m = \beta_c)$ is the Ising critical point. When $\beta_m = 0$ we recover the two degenerate ground states of the low-temperature phase of the pure gauge model with all the plaquettes either in the state $U_{\square} = +1$ (unfrustrated state) or in the state $U_{\square} = -1$ (fully frustrated state).

## 4. Mixed gauge action and stability of the critical points when $\beta_g > 0$

Following Wegner (Wegner 1971, Fradkin and Shenker 1979) we may study the stability of the pure gauge critical point $(\beta_g = \beta_c, \beta_m = 0)$ in the gauge plus matter theory by looking for an effective action at small $\beta_m$. This effective action is obtained by summing over the spin degrees of freedom in $\exp S(\beta_m, \beta_g)$. Then

$$\exp[S_{\text{eff}}(\beta_m, \beta_g)] = \sum_{\{\sigma\}} \exp\left(\beta_m \sum_{(ij)} \sigma_i U_{ij} \sigma_j\right) \exp\left(\beta_g \sum_{\square\square} U_{\square\square}\right) \tag{4.1}$$

may be evaluated near the pure gauge line through a high-temperature expansion of the first exponential:

$$\exp[S_{\mathrm{eff}}(\beta_{\mathrm{m}}, \beta_{\mathrm{g}})] = 2^N (\cosh \beta_{\mathrm{m}})^{2N} \exp\left(\beta_{\mathrm{g}} \sum_{\square\square} U_{\square\square}\right) \sum_{\{g\}} (\tanh \beta_{\mathrm{m}})^{L(g)} \prod_{(ij) \subset g} U_{ij} \tag{4.2}$$

where the last sum is over the Ising graphs $\{g\}$, $L(g)$ is the number of links in $g$ and the product is over the link variables in $g$. For small $\beta_{\mathrm{m}}$ the leading term is 1 corresponding to $L(g)=0$, then come the plaquette graphs with four links around a plaquette, then the window graphs with six links around a window so that

$$\begin{aligned}\exp[S_{\mathrm{eff}}(\beta_{\mathrm{m}}, \beta_{\mathrm{g}})] &\\ &= 2^N (\cosh \beta_{\mathrm{m}})^{2N} \exp\left(\beta_{\mathrm{g}} \sum_{\square\square} U_{\square\square}\right) \\ &\quad \times \left(1 + \tanh^4 \beta_{\mathrm{m}} \sum_{\square} U_{\square} + \tanh^6 \beta_{\mathrm{m}} \sum_{\square\square} U_{\square\square} + \ldots\right).\end{aligned} \tag{4.3}$$

Exponentiating the last bracket, the window correction renormalises $\beta_{\mathrm{g}}$ whereas the plaquette correction introduces a symmetry-breaking term in the effective action:

$$S_{\mathrm{eff}}(\beta_{\mathrm{m}}, \beta_{\mathrm{g}}) = \tanh^4 \beta_{\mathrm{m}} \sum_{\square} U_{\square} + (\beta_{\mathrm{g}} + \tanh^6 \beta_{\mathrm{m}}) \sum_{\square\square} U_{\square\square} + \mathrm{O}(\tanh^8 \beta_{\mathrm{m}}). \tag{4.4}$$

Using the correspondence with the Ising model in an external field, we see that the pure gauge critical point does not give rise to a critical line in the $(\beta_{\mathrm{g}}, \beta_{\mathrm{m}})$ plane (figure 3). The same is true of the Ising critical point on the line $\beta_{\mathrm{g}} = \infty$ by duality.

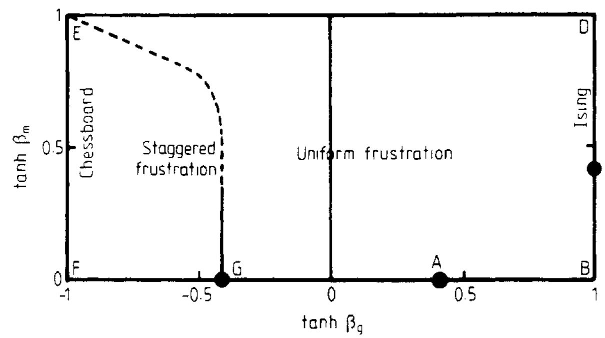


**Figure 3.** Phase diagram of the generalised gauge plus matter theory. A, C and G are Ising critical points. The average frustration increases from zero to one between A and B, remains equal to one along BCDE and vanishes along EFGA. The staggered frustration is maximum along EF and decreases to zero between F and G. The critical line GE separates a phase with staggered frustration from another with uniform frustration.

The line $\beta_{\mathrm{m}} = 0$, $\beta_{\mathrm{g}} > \beta_{\mathrm{c}}$ is rather particular; when $\beta_{\mathrm{m}} = 0$, there are two degenerate ground states with $\langle U_{\square} \rangle$ positive or negative, but for real $\beta_{\mathrm{m}}$ only the state with $\langle U_{\square} \rangle > 0$ may be reached in the limit $\beta_{\mathrm{m}} \to 0_+$ or $\beta_{\mathrm{m}} \to 0_-$ since $\beta_{\mathrm{m}}$ enters the symmetry-breaking term through $\tanh^4 \beta_{\mathrm{m}}$. A first-order transition may occur along this line only in a complex $\beta_{\mathrm{m}}$ field.

## 5. Negative $\beta_g$ sector of the phase diagram

The $\beta_g < 0$ self-dual pure gauge model (equation (2.1)) may be transformed into the $\beta_g > 0$ model through a chessboard transformation under which frustrations change sign on a chessboard sublattice (figure 4). It follows that the ($\beta_g = -\beta_c, \beta_m = 0$) point is also an Ising critical point. When $|\beta_g| > \beta_c$ a non-vanishing average staggered frustration is obtained corresponding to the staggered magnetisation in the Ising model.

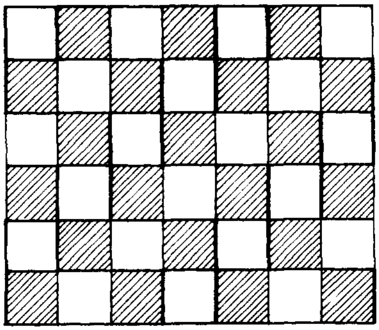

**Figure 4.** Chessboard frustration configuration. In the chessboard Ising model the heavy links are antiferromagnetic.

This state in which the average frustration is zero does not spontaneously break the spin reflection symmetry but only the two sublattices symmetry. As a consequence for small $\beta_m$ the plaquette term in the effective action no longer destroys the transition and, like in the antiferromagnetic Ising model in a uniform external field, we get a second-order Ising critical line in the ($\beta_g$, $\beta_m$) plane. The critical line of the antiferromagnetic Ising model is not known exactly. An approximate expression has been obtained by Müller-Hartmann and Zittartz (1977) for which the interfacial tension associated with particular interface configurations vanishes. It reads

$$\cosh \beta_p = \sinh^2 2\beta_g, \qquad \beta_g < 0, \tag{5.1}$$

with our notations for the mixed action theory. Near the pure gauge line we know from equation (4.4) that

$$\beta_p = \tanh^4 \beta_m \tag{5.2}$$

and combining this with the Müller-Hartmann–Zittartz result we get the approximate critical line near $\beta_m = 0$ (figure 3).

The self-duality of the gauge plus matter theory cannot be used to study the $\beta_g < 0$ sector since it maps the pure gauge line ($\beta_m = 0, \beta_g < 0$) onto the line ($\beta_g = \infty$, $\beta_m$ complex). Anyway we know that when $\beta_g \to -\infty$, the $U_{\square\square}$ must be frozen in the state $-1$ corresponding to two degenerate chessboard configurations for the frustrations (figure 4). Using a gauge where the heavy links in figure 4 are equal to $-1$, we get the chessboard Ising model (André *et al* 1979) which is known to have no transition. The critical line starting at ($\beta_g = -\beta_c$, $\beta_m = 0$) is expected to end at ($\beta_g = -\infty$, $\beta_m = +\infty$) and to remain an Ising critical line since taking into account higher-order terms in the effective action (equation (4.4)) amounts to including irrelevant multispin terms in the antiferromagnetic Ising model and the line $\beta_g = -\infty$ has the same frustration symmetry as the low-temperature phase of the pure gauge theory.

To conclude let us briefly mention that further generalisation of the 2D gauge invariant Ising model involving gauge terms with $n \times 1$ windows ($n > 2$) will have Ising spin formulations with multispin interactions. For $n$ large enough these self-dual models are known to have a first-order transition (Turban 1982, Turban and Debierre 1982, Debierre and Turban 1983, Penson *et al* 1982).

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